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Two Coincidence and Fixed Point Theorems for Hybrid Strict Contractions

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Abstract. In this paper two general coincidence and fixed point theorems for hybrid pairs satisfying an implicit relation are proved.

1. Introduction

Sessa [12] introduced the concept of weakly commuting mappings. Jungck [3] defined the notion of compatible mappings in order to generalize the concept of weak commutativity and showed that weak commuting mappings are compatible but the converse is not true.

In recent years, a number of fixed point theorems have been obtaind by varyous authors by using this notions. Jungck further weakened the notion of compatibility by introducing the notion of weak compatibility [4]. In [5] Jungck and Rhoades further extended weak compatibility.

Pant [8, 9, 10] initiated the study of noncompatible mappings. Singh and Mishra [13] introduced the notion of (I, T)–commutativity.

More recently, Aamri and El-Moutawakil [1] defined a property (E.A.) for self – mappings and obtained some fixed point theorems for such mappings under strict contractive conditions. The class of (E.A.) mappings contain the class of noncompatible maps. Recently, Kamran [6] extend the property (E.A.) for a pair of single and multivalued maps and generalize the notion of (I, T)–commutativity for such pairs. In [6] some coincidence and fixed point theorems for hybrid pairs are obtained, which generalize the results from [1]. Quite recently, O'Regan and Shahzad [11] proved some theorems which generalize the results by Kamran.

2. Preliminaries

Let (X, d) be a metric space. We denote by $CL(X)$ the family of all nonempty closed subsets of X and by $CB(X)$ the family of all nonempty bounded closed subsets of X. Let H be the generalized Hausdorff distance on $CL(X)$. Let $T : (X, d) \to CL(X)$ be a multifunction and $f : (X, d) \to (X, d)$ be a single valued

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mapping. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$. A point $x \in X$ is said to be a coincidence point of f and T if $fx \in Tx$. The set of coincidence points of f and T is denoted by $C(f,T)$. The pair $\{f,T\}$ is called commuting if $fTx = Tfx$ for all $x \in X$, weakly commuting [5] if f and T commute at all points in $C(f,T)$, (I,T)-commuting [2] at $x \in X$ if $fTx \subset Tfx$.

The mappings $f : (X,d) \to (X,d)$ and $T : (X,d) \to CL(X)$ are said to be compatible [7] if $fTx \in CL(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = A \in CL(X)$ and $\lim_{n\to\infty} fx_n = t \in A$.

Therefore the maps $f : (X,d) \to (X,d)$ and $T : X \to CL(X)$ are noncompatible if $fTx \in CL(X)$ for all $x \in X$ and there exist at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = A \in CL(X)$ and $\lim_{n\to\infty} f(x_n) = t \in A$ but $\lim_{n\to\infty} H(fTx_n, Tfx_n) \neq 0$ or non exists.

Definition 1. The mappings $f : (X,d) \to (X,d)$ and $T : (X,d) \to CL(X)$ are said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in CL(X)$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$.

Remark 1. Every noncompatible mappings hybrid pair (f,T) satisfy property (E.A.).

The following theorem is proved in [6].

Theorem 1. *Let f be a self map of the metric space (X,d) and T be a map from X into $CB(X)$ such that:*

(i) *f and T satisfy the property (E.A.);*

(ii) *for all $x \neq y \in X$*

(1)
$$H(Tx,Ty) < \max\{d(fx,fy), [d(fx,Tx)+d(fy,Ty)]/2, [d(fx,Ty)+d(fy,Tx)]/2\}.$$

If $f(X)$ is a closed subset of X then f and T have a coincidence point.

A generalization of Theorem 1 is proved in [11].

Theorem 2. *Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ such that f and T satisfy property (E.A.). Suppose that there exists a continuous functions $\Phi : [0,\infty] \to [0,\infty]$ and continuous functions $\Phi_i : [0,\infty] \to [0,\infty]$, $i = 1,2,\ldots,7$ satisfying $\Phi_i(0) = 0$ for $i = 1,2,4$ and $\Phi(\Phi_i(z)) < z$ for $z > 0$ and $i = 3,5,6,7$ and*

(2)
$$\begin{aligned} H(Tx,Ty) \leq \Phi\Big(\max\Big\{&\Phi_1\big(d(fx,fy)\big), \Phi_2\big(d(fx,Tx)\big),\\ &\Phi_3\big(d(fy,Ty)\big), \Phi_4\big(d(fy,Tx)\big), \Phi_5(d(fx,Ty)),\\ &\Phi_6\big(d(fx,fy)+d(fx,Tx)+d(fy,Tx)+d(fy,Ty)\big),\\ &\Phi_7\big(d(fx,fy)+d(fx,Tx)+d(fy,Tx)+d(fx,Ty)\big)\Big\}\Big) \end{aligned}$$

for all $x,y \in X$.

If $f(X)$ is closed, then $C(f,T) \neq \Phi$.

The purpose of this paper is to generalize Theorems 1 and 2 for hybrid pairs which satisfies implicit relations.

3. Implicit relations.

Let F_6 be the set of all continuous functions $F : R_+^6 \to R$ satisfying the following conditions:

(F1) F is nondecreasing in variable t_1;
(F2) $F(t, 0, 0, t, 0, t) \leq 0$ implies $t = 0$.

Example 1. $F(t_1, \ldots, t_6) = t_1 - \Phi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$, where $\Phi : R_+ \to R$ satisfying conditions:

a) Φ is continuous;
b) Φ is nonincreasing;
c) $0 < \Phi(t) \leq t$, $\forall t > 0$.

(F1): Obviously;
(F2): If $F(t, 0, 0, t, 0, t) = t - \Phi(\frac{t}{2}) \leq 0$, then $t \leq \Phi(\frac{t}{2}) \leq \frac{t}{2}$. Hence $t = 0$.

Example 2. $F(t_1, \ldots, t_6) = t_1^3 - \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1+t_1+t_2}$.

(F1): Obviously.
(F2): If $F(t, 0, 0, t, 0, t) = t^3 \leq 0$, then t=0.

Example 3. $F(t_1, \ldots, t_6) = t_1^2 - (at_2^2 + bt_3t_4 + ct_5t_6)$ where $a, b, c \geq 0$.

(F1): Obviously.
(F2): If $F(t, 0, 0, t, 0, t) = t^2 \leq 0$, then $t = 0$.

Example 4. $F(t_1, \ldots, t_6) = t_1 - G\Big[\max\{g_1(t_2), g_2(t_3), g_3(t_4), g_4(t_5)g_5(t_6), g_6(t_2+t_3+t_4+t_5), g_7(t_2+t_3+t_5+t_6)\}\Big]$ where $G : R_+ \to R_+$ is continuous, $g_i : R_+ \to R_+$, $i = 1, 2, 3, \ldots, 7$, satisfying $g_i(0) = 0$ for $i = 1, 2, 4$ and $G(g_i(t)) < t$ for $t > 0$ and $i = 3, 5, 6, 7$.

(F1): Obviously.
(F2): Let $F(t, 0, 0, t, 0, t) =$
$t - G\Big(\max\{g_1(0), g_2(0), g_3(t), g_4(0), g_5(t)g_6(t), g_7(t))\}\Big) \leq 0.$
This implies $t \leq G\{\max[g_3(t), g_5(t), g_6(t), g_7(t)]\}$
Supose that $g_3(t) = \max\{g_3(t), g_5(t), g_6(t), g_7(t)\}$.
Therefore $t \leq G(g_3(t)) < t$, a contradiction. One can obtain a contradiction in the other cases in a similar fashion. Therefore $t = 0$.

4. Main results

Theorem 3. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ such that f and T satisfy the property (E.A.). If*

$$(3) \qquad F(H(Tx, Ty), d(fx, fy), d(fx, Tx), d(fy, Ty), d(fy, Tx), d(fx, Ty)) < 0$$

for all $x, y \in X$, $F \in F_6$ and $f(X)$ is closed, then $C(f, T) \neq \phi$.

Proof. Since f and T satisfy property (E.A.), there exists a sequence $\{x_n\}$ in X, $t \in X$ and $A \in CL(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Since $f(X)$ is closed there exists $a \in X$ such that

$$\lim_{n\to\infty} fx_n = fa,$$

hence $fa \in A$. We claim that $fa \in Ta$.

If not, then we have by (3) that

$$F\Big(H(Tx_n, Ta), d(fx_n fa), d(fx_n, Tx_n), d(fx, Ta), d(fa, Tx_n), d(fx_n Ta)\Big) < 0.$$

Letting n tend to infinity we obtain:

$$F(H(A, Ta), d(fa, fa), d(t, A), d(fa, Ta), d(fa, A), d(fa, Ta) \le 0.$$

Since $f(a) \in A$ and (F1) we obtain:

$$F(d(fa, Ta), 0, 0, d(fa, Ta), 0, d(fa, Ta)) \le 0.$$

which implies by (F2) that $d(fa, Ta) = 0$. Therefore $fa \in Ta$. □

Corollary 1. *Theorem 1.*

Proof. The proof follows by Theorem 3 and Ex 1 for $\Phi(t) = t$. □

Corollary 2. *Theorem 2.*

Proof. The proof follows by Theorem 3 and Ex 4. □

Since noncompatible hybrid pair (f, T) satisfy property (E.A.) we obtain the following:

Corollary 3. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ such that (f, T) is noncompatible, the inequality (3) holds for all $x, y \in X$, $F \in F_6$ and $f(X)$ is closed, then $C(f, T) \neq \phi$.*

Remark 2. By Corollary 3 and Ex.1 for $\Phi(t) = t$ we obtain Corollary 3.6 [6].

Definition 2. Let $T : X \to CL(X)$. The mapping $f : X \to X$ is said to be T-weakly commuting at $x \in X$ [6] if $ffx \in Tfx$.

Here we remark that for hybrid pair (f, T), (I, T)-commuting at the coincidence points implies that f is T-weakly commuting, but the converse is not true in general (Ex.3-8,[6]).

Theorem 4. *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ such that f and T satisfy the property (E.A.), the inequality (3) holds for all $x, y \in X$, $F \in F_6$ and $f(X)$ is closed. If one of the following condition holds:*

(i) *f is continuous, T is closed, f is (I, T)-commuting at points in $C(f, T)$ and $\lim_{n\to\infty} f^n a$ exists for $a \in C(f, T)$;*

(ii) *f is T weakly commuting at a and $ffa = fa$ for any $a \in C(f, T)$, then f and T have a common fixed point.*

Proof.

(i) The proof is similar to the proof of (i) from [11], Theorem 2.5.
(ii) If (ii) holds, then we have $ffa \in Tfa$, since f and T are weakly commuting. Consequently we have $fa = ffa \in Tfa$. Therefore fa is a common fixed point of f and T. □

Remark 3. By Theorem 4 and Ex.1 for $\Phi(t) = t$ we obtain Theorem 3.10 [6]. By Theorem 4 and Ex.4 we obtain Theorem 2.5 [11].

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